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LETTER TO THE EDITOR

Operator content of $c = 1$ conformal Z_n field theories

P Suranyi†‡

Physikalisches Institut, University of Bonn, Nussalee 12, D-5300 Bonn, Federal Republic of Germany

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Abstract. It is shown that the operator content of frustrated sectors of $c = 1$ Z_n invariant conformal field theories (including the $n \to \infty$ limit, the XXY model) can be obtained from the field theory of a free massless boson with appropriate boundary conditions. It is also shown that, for frustrated boundary conditions, the requirement of modular invariance is substituted by definite transformation properties of the partition function under the modular group, which are satisfied by the partition function constructed in this letter.

Cardy (1986a) has shown that the operator content of systems of the discrete series of $c < 1$ (where c is the conformal anomaly) two-dimensional conformal field theories (CFT) (Belavin *et al* 1984, Friedan *et al* 1984) is fixed by modular invariance (MI) of the partition function. Furthermore, Itzykson and Zuber (1986), Capelli *et al* (1987) and Gepner (1987) were able to use MI to classify all $c < 1$ CFT of the discrete series.

In subsequent work Cardy (1986b) and Zuber (1986) investigated the operator content of some Z_n invariant $c < 1$ CFT with frustrated boundary conditions and/or partition functions restricted to sectors of definite values of the Z_n charge. Again, using modular transformation properties these authors were able to confirm the operator content of various sectors of the Potts model, found earlier using numerical methods by von Gehlen and Rittenberg (1986).

In the present letter the operator content of $c = 1$ Z_n invariant CFT is investigated in the case of frustrated boundary conditions. In a recent paper von Gehlen *et al* (1987) were able to find the operator content of these theories by using a combination of numerical and analytical tools. In the sectors with frustrated boundary conditions, defined below, they found an infinite series of primary fields (all with unit multiplicity) labelled by the unrestricted integers l_1 and l_2, having conformal dimensions

$$(\Delta^{Q\tilde{Q}}_{l_1 l_2}, \bar{\Delta}^{Q\tilde{Q}}_{l_1 l_2}) = \left(\frac{[Q + nl_1 + g(\tilde{Q} + nl_2)]^2}{4ng}, \frac{[Q + nl_1 - g(\tilde{Q} + nl_2)]^2}{4gn}\right) \tag{1}$$

where $\tilde{Q} = 0, 1, 2, \ldots, n-1$ defines the frustration (or twist) in the boundary condition of the quantum chain and Q is the eigenvalue of the Z_n charge operator $\hat{Q}$. g is a positive constant, related to temperature. $\tilde{Q}$ is defined as follows: if O_i denotes operators belonging to the ith site of a quantum chain of length L, then we identify

$$O_{L+1} = \Sigma^{\tilde{Q}} O_1 \Sigma^{-\tilde{Q}} \tag{2}$$

† Present address: Blackett Laboratory, Imperial College of Science and Technology, London SW7 2BZ.
‡ Permanent address: Department of Physics, University of Cincinnati, Cincinnati, OH 45221, USA.

where

$$\Sigma = \exp(\mathrm{i}\hat{Q}2\pi/n). \tag{3}$$

The ends of the line of critical points of the $c=1$ Z_n system for $n>4$ are located at $g=4/n$ and $g=n/4$, where the $Q=\tilde{Q}=0$, $l_1=0$, $l_2=1$ and $l_1=1$, $l_2=0$ operators become marginal, respectively. The self-duality point is $g=1$.

For periodic boundary conditions ($\tilde{Q}=0$), and when one sums over Q, the operator content becomes

$$(\Delta_{l_1 l_2}, \bar{\Delta}_{l_1 l_2}) = \left(\frac{[l_1+ngl_2]^2}{4ng}, \frac{[l_1-ngl_2]^2}{4ng}\right) \tag{4}$$

exactly the dimensions of the Gaussian model, compactified on a circle of radius R, where R is identified as $R=(2ng)^{1/2}$ or $R=(2/ng)^{1/2}$ (Di Francesco *et al* 1987, Dijkgraaf *et al* 1987). Multicritical points of the Gaussian model are at $R=\sqrt{2}$ and $R=1/\sqrt{2}$ in agreement with the second choice for R. This fact already indicates a relation between Z_n symmetric models and compactified Gaussian models. In what follows it will be shown that this relation runs substantially deeper.

First, doubly frustrated partition functions for a quantum system of periodicity L_1 (L_1 is complex) and boundary condition $\tilde{Q}_1$ are introduced as

$$Z(\tilde{Q}_2, \tilde{Q}_1; L_2, L_1) = \mathrm{Tr}[\Sigma^{\tilde{Q}_2} \exp(-H^{\tilde{Q}_1}\,\mathrm{Re}\,L_2 - \mathrm{i}P^{\tilde{Q}_1}\,\mathrm{Im}\,L_2)] \tag{5}$$

where $H^{\tilde{Q}_1}$ and $P^{\tilde{Q}_1}$ are the Hamiltonian and the momentum operator for the quantum chain, respectively. The physical requirement of symmetry for the exchange of boundaries implies (the S generator of the modular group, corresponding to $\tau \to -1/\tau$)

$$Z(\tilde{Q}_2, \tilde{Q}_1; L_2, L_1) \equiv Z^{\tilde{Q}_2\tilde{Q}_1}(\tau) = Z^{-\tilde{Q}_1\tilde{Q}_2}(-1/\tau) = SZ^{-\tilde{Q}_1\tilde{Q}_2}(\tau) \tag{6}$$

where τ is the conformal ratio, $\tau = \mathrm{i}L_2/L_1$.

Let us construct a doubly frustrated partition function for a massless boson in the functional integral form, following the methods of Itzykson and Zuber (1986) and Di Francesco *et al* (1987). The action for the boson field $\Phi(z, \bar{z})$ on a complex torus is written as

$$S = \frac{ng}{4\pi}\int \mathrm{d}z\,\mathrm{d}\bar{z}\,\partial_z\Phi\partial_{\bar{z}}\Phi \tag{7}$$

where $z=x+\mathrm{i}y$, $z=x-\mathrm{i}y$ and the scale factor ng was chosen appropriately.

We split $\Phi(z, \bar{z})$ into two parts:

$$\Phi(z, \bar{z}) = \phi(z, \bar{z}) + \phi_{\mathrm{cl}}(z, \bar{z}) \tag{8}$$

such that ϕ satisfies periodic boundary conditions and ϕ_{cl} satisfies the Laplace equation and frustrated boundary conditions:

$$\phi_{\mathrm{cl}}(z, \bar{z}) = 2\pi\{[(m_1+\tilde{Q}_1/n)k_1 + (m_2+\tilde{Q}_2/n)k_2]z + \mathrm{cc}\} \tag{9}$$

where k_1 and k_2 are complex wavenumbers, m_1, m_2, $\tilde{Q}_1$, $\tilde{Q}_2$ and n are integers and cc means complex conjugation, i.e. $k_1 \to \bar{k}_1$, $k_2 \to \bar{k}_2$, $z \to \bar{z}$. The complex periods of the torus ω_1 and ω_2 are related to k_1 and k_2 as follows:

$$k^1 = -\mathrm{i}\omega_2/A \qquad k^2 = \mathrm{i}\omega_1/A \tag{10}$$

where $A = \mathrm{Im}(\omega_2\bar{\omega}_1)$ is the 'area' of the torus. The quantities k^i and ω_j satisfy the relation

$$\mathrm{Re}(k^i\omega_j) = \delta^i_j. \tag{11}$$

It follows from (11) that

$$\phi_{cl}(z+\omega_i, \bar{z}+\bar{\omega}_i) = \phi_{cl}(z, \bar{z}) + 2\pi(m_i + \tilde{Q}_i/n) \tag{12}$$

and of course a similar equation is valid for $\Phi(z, \bar{z})$ as well. If Φ is regarded as a phase, then the field $\exp(i\Phi)$ satisfies the required frustrated boundary conditions. The partition function built from the action (7) obviously satisfies symmetry condition (6). It has the following form:

$$Z^{\tilde{Q}_1\tilde{Q}_2} \sim \int \mathrm{d}\phi \sum_{m_1 m_2} \exp\left(-\frac{ng}{4\pi}\int \mathrm{d}z\,\mathrm{d}\bar{z}[\partial_z\phi\partial_{\bar{z}}\phi + \partial_z\phi_{cl}\partial_{\bar{z}}\phi_{cl}]\right). \tag{13}$$

After straightforward calculations one obtains

$$Z^{\tilde{Q}_1\tilde{Q}_2} = Z_0 \sum_{m_1 m_2} \exp\left\{-gn\pi\left[\left(m_1+\frac{\tilde{Q}_1}{n}\right)^2\frac{1}{\tau_I} + \left(m_2+\frac{\tilde{Q}_2}{n}\right)^2\frac{\tau_R^2+\tau_I^2}{\tau_I}\right.\right.$$
$$\left.\left. -2\left(\frac{m_1+\tilde{Q}_1}{n}\right)\left(\frac{m_2+\tilde{Q}_2}{n}\right)\frac{\tau_R}{\tau_I}\right]\right\} \tag{14}$$

where $\tau = \tau_R + i\tau_I$ and Z_0 is the partition function for the ground-state contribution:

$$Z_0 = (q\bar{q})^{-1/24}\pi(q)\pi(\bar{q}) \tag{15}$$

where $\pi(q)$ is the Dedekind function.

Poisson summation over m_1 and m_2 (the corresponding integer variables are m and l_2) gives

$$Z^{\tilde{Q}_1\tilde{Q}_2} = Z_0 \sum_{l_2 m} \exp\{-2\pi i m\tilde{Q}_1/n - \tau_I\pi[m^2/gn + gn(l_2+\tilde{Q}_2/n)^2]$$
$$-2i\pi m\tau_R(l_2+\tilde{Q}_2/n)\}. \tag{16}$$

Substituting $m = l_1 n - Q$ results in the expression

$$Z^{\tilde{Q}_1\tilde{Q}_2} = Z_0 \sum_{l_1 l_2}\sum_{Q} \exp\left(2i\pi\frac{Q\tilde{Q}_1}{n} + \frac{2i\pi\tau}{4gn}[l_1 n + Q + g(l_2 n + \tilde{Q}_2)]^2\right.$$
$$\left. -\frac{2i\pi\bar{\tau}}{4gn}[l_1 n + Q - g(l_2 n + \tilde{Q}_2)]^2\right) \tag{17}$$

in agreement with the operator content found by von Gehlen *et al* (1987) in the appropriate sectors. We can conclude that $c = 1$ Z_n CFT is equivalent to a field theory of massless frustrated bosons.

The above results can be generalised to the $n \to \infty$ limit (keeping $\tilde{g} = gn$ fixed) as well. One only has to substitute $\tilde{Q}_i/n$ by s_i in (9), (12), (14) and (16), after which one substitutes $m = -Q$ and $l_2 = l$ in (16) to arrive at the spectrum of primary fields, each with unit multiplicity, labelled by the single integer l:

$$(\Delta_l^{Qs}, \bar{\Delta}_l^{Qs}) = \left(\frac{[Q+\tilde{g}(l+s)]^2}{4\tilde{g}}, \frac{[Q-\tilde{g}(l+s)]^2}{4\tilde{g}}\right) \tag{18}$$

where, of course, Q is the Z_∞ charge and $0 \leqslant s < 1$ defines a continuous set of boundary conditions. This operator content is in exact agreement with that of the XXY model as obtained by von Gehlen *et al* (1987).

Notice that the doubly frustrated partion function has simple transformation properties not only under generator S of the modular group, as given by (6), but the other generator, T, as well, corresponding to the transformation $\tau \to \tau + 1$. Performing such a transformation on partition function (16) immediately gives

$$TZ^{\tilde{Q}_1 \tilde{Q}_2}(\tau) = Z^{\tilde{Q}_1 \tilde{Q}_2}(\tau + 1) = Z^{\tilde{Q}_1 + \tilde{Q}_2, \tilde{Q}_2}(\tau) \tag{19}$$

where $\tilde{Q}_1 + \tilde{Q}_2$ is taken mod n.

Equations (6) and (19) imply that the doubly frustrated partition function, instead of being invariant, transforms as a well defined (reducible) representation of the modular group (Dixon *et al* 1986). It is easy to see that transformation property (19) of the partition function is a general feature in arbitrary CFT with Z_n symmetry. This fact had already been recognised by Fradkin and Kadanoff (1980) and it is a consequence of that that the $\tau \to \tau + 1$ transformation corresponds to the circumnavigation of a toroidal or cylindrical system. On the one hand, this corresponds to a rotation by 2π and as such introduces a phase factor of $2\pi s = 2\pi(\Delta - \bar{\Delta})$, where s is the spin. On the other hand, in a sector of definite charge and boundary conditions it introduces a factor of $2\pi Q\tilde{Q}/n$ as implied by (2) and (3). The equality of these two phases (mod 2π) and the definition of $Z^{\tilde{Q}_1 \tilde{Q}_2}$ leads to (19).

We conjecture that the modular covariance of $Z^{\tilde{Q}_1 \tilde{Q}_2}$ ((6) and (19)), together with the constraint that the coefficients in the character expansion are integers multiplied by the phases $\exp(2\pi i Q\tilde{Q}/n)$ fixes the possible Z_n invariant theories at every value of c, much like modular invariance does for periodic boundary conditions. We were not able to find any other $c = 1$ Z_n invariant CFT with appropriate modular transformation properties, than that given by (17).

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